DELAY-DEPENDENT STABILITY CRITERIA OF LINEAR DISCRETE SYSTEMS WITH MULTIPLE TIME DELAY*

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This paper offers necessary and sufficient condition for delay-dependent asymptotic stability of linear discrete systems with multiple time delay. The criterion is derived by Lyapunov’s direct method and two matrix equations have been obtained: discrete Lyapunov matrix equation for nondelay systems and polynomial matrix equation. Obtained stability condition does not possess conservatism but require some numerical computations. However, if the dominant solvent of the polynomial matrix equation can be computed by Traub’s or Bernoulli’s algorithm, then smaller number of computations are obtained compared with a traditional stability procedure based on eigenvalues of matrix of augmented system. Numerical computations are performed to illustrate the results obtained.

Key words: discrete systems, time delay, Lyapunov stability, delay-dependent stability

INTRODUCTION

Time delay is very often encountered in various technical systems, such as electric, pneumatic and hydraulic networks, chemical processes, long transmission lines, etc. The existence of pure time lag, regardless if it is present in the control or/and the state, may cause undesirable system transient response, or even instability.

During the last three decades, the problem of stability analysis of time delay systems has received considerable attention and many papers dealing with this problem have appeared. In the literature, various stability techniques have been utilized to derive stability criteria for time delay systems by many researchers. The developed stability criteria are classified often into two categories according to their dependence on the

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size of the delay: delay-dependent and delay-independent stability criteria. It has been shown that delay-dependent stability conditions that take into account the size of delays, are generally less conservative than delay-independent ones which do not include any information on the size of delays.

Further, the delay-dependent stability conditions can be classified into two classes: frequency-domain (which are suitable for systems with a small number of heterogeneous delays) and time-domain approaches (for systems with a many heterogeneous delays). The second approach is based on the comparison principle based techniques for functional differential equations [1, 2] or the Lyapunov stability approach with the Krasovskii and Razumikhin methods [3, 4]. The stability problem is thus reduced to one of finding solutions to Lyapunov [5] or Riccati equations [6] solving linear matrix inequalities (LMIs) [7] or analyzing eigenvalue distribution of appropriate finite-dimensional matrices [8].

It is well-known [9] that the choice of an appropriate Lyapunov–Krasovskii functional is crucial for deriving stability conditions. The general form of this functional leads to a complicated system of partial differential equations [10]. Special forms of Lyapunov–Krasovskii functionals lead to simpler delay-independent [7, 9, 11] and (less conservative) delay-dependent conditions [9, 12, 13].

In the past few years, there have been various approaches to reduce the conservatism of delay-dependent conditions by using new bounding for cross terms (Park’s and similarly inequalities) [14, 15] or choosing new Lyapunov–Krasovskii functional and model transformation. However, the model transformation may introduce additional dynamics [16]. In [17] it is shown that descriptor transformation leads to a system which is equivalent to the original one, does not depend on additional assumptions for stability of the transformed system and requires bounding of fewer cross-terms. In [18] both the descriptor system approach and the bounding technique using by [15] are utilized and the delay-dependent stability results are performed. The derived stability criteria have been demonstrated to be less conservative than existing ones in the literature.

The majority of stability conditions in the literature available, of both continual and discrete time delay systems, are sufficient conditions. Only a small number of works provide both necessary and sufficient conditions [19-24] which are in their nature mainly dependent of time delay. These conditions do not possess conservatism but often require more complex numerical computations. In our paper we represent some necessary and sufficient stability conditions.

Basic inspiration for our investigation is based on paper [19] dealing with the stability of continuous linear time delay systems. In our paper we present delay-dependent stability criteria, based on advanced computational procedures, for linear discrete time delay systems. These stability criteria are express in form necessary and sufficient conditions.

Throughout this paper we use the following notation. \( \mathbb{R} \) and \( \mathbb{C} \) denote real (complex) vector space or the set of real (complex) numbers, \( \mathbb{N}_0 \) denotes the set of all non-negative integers, \( \lambda^* \) means conjugate of \( \lambda \in \mathbb{C} \) and \( F^* \) conjugate transpose of matrix \( F \in \mathbb{C}^{n \times n} \). \( \Re(s) \) is the real part of \( s \in \mathbb{C} \). The superscript T denotes transposition. For
real matrix $F$ the notation $F > 0$ means that the matrix $F$ is positive definite. $\lambda_i(F)$ is the eigenvalue of matrix $F$. Spectrum of matrix $F$ is denoted with $\sigma(F)$ and spectral radius with $\rho(F)$.

**MODEL DESCRIPTION AND PRELIMINARIES**

A linear, discrete time-delay system with multiple delay can be represented by the following difference equation

$$x(k+1) = \sum_{j=0}^{N} A_j x(k-h_j), 0 = h_0 < h_1 < \cdots < h_N$$

(1)

with an associated function of initial state

$$x(\theta) = \psi(\theta), \theta \in \{-h_N, -h_N+1, \ldots, 0\}$$

(2)

The equation (1) is referred to as homogenous or the unforced state equation. Vector $x(k) \in \mathbb{R}^n$ is a state vector, $A_j \in \mathbb{R}^{n \times n}$ are constant matrices and time delays are expressed by integers $h_j \in \mathbb{T}^+$. System (1) can be expressed with the following representation without delay (augmented system) [10].

$$\hat{x}(k+1) = \hat{A}\hat{x}(k), \quad \hat{x}(k) = [x^T(k-h_N) x^T(k-h_N+1) \cdots x^T(k)] \in \mathbb{R}^N, N \triangleq n(h_N+1)$$

$$\hat{A} = \begin{bmatrix} 0 & I_{nh_N} & \cdots & 0 \\ -A_N & -A_{N} \cdots & -A_0 \\ \end{bmatrix} \in \mathbb{R}^{N \times N}$$

(3)

**Definition 1.** Characteristic polynomial of system (1) [10] is given with:

$$f(\lambda) = \det M(\lambda) = \sum_{j=0}^{N} a_j \lambda^j, a_j \in \mathbb{R}, \quad M(\lambda) = I_n \lambda^{h_N+1} - \sum_{j=0}^{N} A_j \lambda^{h_N-h_j}$$

(4)

**Definition 2.** Denote with

$$\Omega \triangleq \{ \lambda \ | \ f(\lambda) = 0 \} = \lambda(\hat{A})$$

(5)

the set of all roots of the characteristic equation of system (1). A root $\lambda \in \Omega$ with maximal module

$$\lambda_m \in \Omega : |\lambda_m| = \max \left| \lambda \left( \hat{A} \right) \right|$$

(6)

let us call maximal eigenvalue of system (1).

**MAIN RESULTS**

If scalar variable $\lambda$ in the characteristic polynomial is replaced by matrix $X \in \mathbb{R}^{n \times n}$ the following monic matrix polynomial is obtained
\[ M(X) = X^{h_n+1} - \sum_{j=0}^{N} X^{h_n-h_j} A_j \]  

**Definition 3.** If \( M(S) = 0 \)

we say that \( S \) is a solvent of (8).

**Lemma 1.** The matrix \( M(\lambda) \) can be factorized in the following way

\[ M(\lambda) = (\lambda I_n - S) \left( \sum_{i=0}^{h_n} \lambda^{h_n-i} S^i - \sum_{j=0}^{N-1} \sum_{i=0}^{h_n-h_j} \lambda^{h_n-h_j-i} X^i S^j A_j \right) \]

**Proof.**

\[ M(\lambda) - M(X) = \lambda^{h_n+1} I_n - X^{h_n+1} - \sum_{j=0}^{N-1} \left( \lambda^{h_n-h_j} I_n - X^{h_n-h_j} \right) A_j \]

\[ = (\lambda I_n - X) \left( \sum_{i=0}^{h_n} \lambda^{h_n-i} X^i - \sum_{j=0}^{N-1} \lambda^{h_n-h_j-i} X^i S^j A_j \right) \]

If \( S \) is a solvent of \( M(X) \), from (10) follows (9).

**Lemma 2.** For solvent \( R \) defined by (8) hold

\[ \lambda(S) \subset \Omega \]

**Proof.** From Lemma 1 follows \( f(S) = 0 \). Therefore, the characteristic polynomial \( f(\lambda) \) of system (1) is *annihilating polynomial* for solvent \( S \). So \( \lambda(S) \subset \Omega \).

For the needs stability of system (1), only the maximal solvent(s) of (8) is (are) usable, whose spectrums contain maximal eigenvalue \( \lambda_m \).

**Definition 4.** Every solvent \( S_m \) of (8) whose spectrum \( \sigma(S_m) \) contains maximal eigenvalue \( \lambda_m \) of \( \Omega \) is a *maximal solvent* of (8).

A special case of maximal solvent is the so called *dominant solvent* [25, 26] which can be computed in a simple way by Bernoulii or Traub algorithm.

**Definition 5.** [25, 26] Matrix \( A \) dominates matrix \( B \) if all the eigenvalues of \( A \) are greater, in modulus, then those of \( B \). In particular, if the solvent \( S_1 \) of (8) dominates over all remaining solvents of (8), we say that \( S_1 \) is a *dominant solvent* of (8).

**Remark 1.** The number of maximal solvents can be greater than one. Dominant solvent is at the same time maximal solvent too. The dominant solvent \( S_1 \) of (8), under certain conditions, can be determined by the *Traub* and *Bernoulli algorithm* [26].

**Theorem 1.** Suppose that there exists at least one maximal solvent of (8) and let \( S_m \) denote one of them. Then, linear discrete time delay system (1) is asymptotically stable if and only if for any given matrix \( Q = Q^* > 0 \) there exists Hermitian matrix \( P = P^* > 0 \) that satisfies following Lyapunov equation

\[ S_m^* P S_m - P = -Q \]
Proof. \textit{(Sufficient condition)} Define the following vector of discrete functions

\[ z(x_k) = x(k) + \sum_{j=1}^{h_j} \sum_{l=1}^{h_j} T_j(l) x(k-l), \quad x_k = x(k + \theta), \quad \theta \in \{-h, -h+1, \ldots, 0\} \tag{13} \]

where, \( T_j(k) \in \mathbb{C}^{n \times n} \) is, in general, some time varying discrete matrix function. For Lyapunov functional we adopt

\[ V(x_k) = \bar{z}^*(x_k) P z(x_k), \quad P = P^* > 0 \tag{14} \]

The forward difference of (14), along the solutions of system (1) is

\[ \Delta V(x_k) = \Delta \bar{z}^*(x_k) P z(x_k) + \bar{z}^*(x_k) P \Delta z(x_k) + \Delta \bar{z}^*(x_k) P \Delta z(x_k) \tag{15} \]

A difference of \( \Delta z(x_k) \) can be determined in the following manner

\[ \Delta z(x_k) = \Delta x(k) + \sum_{j=1}^{N} \sum_{l=1}^{h_j} T_j(l) \Delta x(k-l) \tag{16} \]

\[ \sum_{l=1}^{h_j} T_j(l) \Delta x(k-l) = T_j(1) x(k) - T_j(h_j) x(k-h_j) + \sum_{l=1}^{h_j-1} \Delta T_j(l) \Delta x(k-l) \tag{17} \]

\[ \Delta z(x_k) = \left( A_0 - I_n + \sum_{j=1}^{N} T_j(1) \right) x(k) + \sum_{j=1}^{N} \left( A_j - T_j(h_j) \right) x(k-h_j) \tag{18} \]

Define a new matrix \( S \) by

\[ S \bigtriangleup A_0 + \sum_{j=1}^{N} T_j(1) \tag{19} \]

If

\[ A_j - T_j(h_j) = \Delta T_j(h_j) \tag{20} \]

then \( \Delta z(x_k) \) has a form

\[ \Delta z(x_k) = (S - I_n) x(k) + \sum_{j=1}^{N} \sum_{l=1}^{h_j} \Delta T_j(l) x(k-l) \tag{21} \]

If one adopts

\[ \Delta T_j(l) = (S - I_n) T_j(l), \quad l = 1, 2, \ldots, h_j \tag{22} \]

then \( \Delta z(x_k) \) becomes

\[ \Delta z(x_k) = (S - I_n) \left( x(k) + \sum_{j=1}^{N} \sum_{l=1}^{h_j} T_j(l) x(k-l) \right) = (S - I_n) z(x_k) \tag{23} \]

Therefore, (15) becomes

\[ \Delta V(x_k) = \bar{z}^*(x_k) \left( S^* P S - P \right) z(x_k) \tag{24} \]

It is obvious that if the following equation is satisfied

\[ S^* P S - P = -Q, \quad Q = Q^* > 0 \tag{25} \]
then $\Delta V(x_k) < 0, x_k \neq 0$.

In the Lyapunov matrix equation (25), of all possible solvents $S$ of (8) only one of maximal solvents $S_m$ is of importance, for it is the only one that contains maximal eigenvalue $\lambda_m \in \Omega$, which has dominant influence on the stability of the system. If solvent which is not maximal is integrated into Lyapunov equation, it may happen that there will exist positive definite solution of Lyapunov matrix equation (25), although the system is not stable.

(Necessary condition) If the system (1) is asymptotically stable then all roots $\lambda_i \in \Omega$ are located within unit circle. Since $\sigma(S_m) \subset \Omega$, follows $\rho(S_m) < 1$, so the positive definite solution of Lyapunov matrix equation (12) exists.

If solvent $S$ ($S_m$) exists, it can be determined as follows. From (22) it can be showed, that

$$T_j(l+1) = ST_j(l), \quad T_j(h_j) = S^{h_j}T_j(1), \quad 1 \leq j \leq N$$

(26)

while

$$A_j - T_j(h_j) = \Delta T_j(h_j) = (S-I_n)T_j(h_j)S, \quad ST_j(h_j) = A_j$$

(27)

follows from (20). Using and combining of two last equations, one can get the following system of matrix equations

$$S^{h_j}T_j(1) = A_j, \quad 1 \leq j \leq N$$

(28)

Now it is possible, using (19), to determine basic matrix $S$ and unknown matrices $T_j(1)$. Moreover

$$S^{h_y} : 0 = S - \left[ A_0 + \sum_{j=1}^{N} T_j(1) \right] = S^{h_y+1} - S^{h_y}A_0 - \sum_{j=1}^{N} S^{h_y-h_j} \left[ S^{h_j}T_j(1) \right]$$

(29)

Corollary 1. Suppose that there exists at least one maximal solvent of (8) and let $S_m$ denote one of them. Then, system (1) is asymptotically stable if and only if $\rho(S_m) < 1$.

Proof. Follows directly from Theorem 1.

Corollary 2. Suppose that there exists dominant left solvent $S_1$ of (8). Then, system (1) is asymptotically stable if and only if $\rho(S_1) < 1$.

Proof. Follows directly from Corollary 1, since dominant solution is, at the same time, maximal solvent.

Remark 2. In the case when dominant solvent $S_1$ may be computed by Traub or Bernoulli algorithm, Corollary 2 represents a quite simple method.
NUMERICAL EXAMPLES

Example 1. Let us consider linear discrete system with delayed state (1) with
\[
A_0 = \begin{bmatrix} 0.01 & 0 \\ 0.2 & 0.2 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0.2 & 0 \\ 0.1 & 0.5 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0.5 & 0 \\ 0.1 & 0.2 \end{bmatrix}, \quad h_1 = 2, \quad h_2 = 10
\]
From (8) follows
\[
S^{11} - S^{10} A_0 - S^h A_1 - A_2 = 0
\]
In this case the maximal solvent coincide with dominant solvent
\[
S_m = S_1 = \begin{bmatrix} 0.9621 & 0.0347 \\ 0 & 0.9768 \end{bmatrix}
\]
The same solution can be obtained using Traub or Bernoulli algorithm. So \(\rho(S_1) = 0.9768 < 1\) and based on Theorem 1, Corollary 1 or 2, the system is asymptotically stable. The same conclusion can be obtained using augmented system with matrix \(\hat{A} \in \mathbb{R}^{22 \times 22}\).

CONCLUSION

In this paper, we have established new, necessary and sufficient conditions for the asymptotic stability of a particular class of linear discrete time delay systems. The time-dependent criteria are derived by Lyapunov’s direct method and are exclusively based on the maximal and dominant solvents of particular matrix polynomial equation. In the case when dominant solvent exist, the proposed method for system stability investigation can be easily applied.
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IZVOD

KRITERIJUMI STABILNOSTI ZAVISNI OD KAŠNJENJA ZA LINEARNE DISKRETNJE SISTEME SA VIŠE KAŠNJENJA
(Naučni rad)
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